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Diffraction of Pulses by a Circular Cylinder

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Abstract

This paper deals with the diffraction of a sound pulse due to an instantaneous line source by a parallel, rigid, fixed circular cylinder. This problem can also be interpreted in electromagnetic terms. The diffracted field is resolved into terms representing diffracted pulses which have encircled the cylinder a number of times in either sense, and these in turn are expressed as a series of 'propagation modes' with the same pulse fronts. Approximations valid near the front of a mode are then obtained. The corresponding results for an incident plane pulse are derived by a limiting process. It is found that a diffracted pulse has an essential singularity of a definite type at its fronts, similar to the initial stages of diffusion, and that the pressure rises very gradually at first. But after time intervals of the order of one-third to one-half of the radius of the cylinder divided by the velocity of sound, there seems to be no appreciable diffraction effect.

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Introduction

The pressure field behind the front of a sound pulse can be discussed quite easily if it is assumed that it is a regular function of the time elapsed since the arrival of the pulse. The front itself is propagated according to the laws of geometrical optics; the square of the pressure jump at the front obeys the intensity law of geometrical optics [1,3]; finally, the derivatives of the pressure with respect to time at the front can be calculated recursively, so that the pressure can be expanded in an ascending series of powers of the time counted from the arrival of the front.

When a sound pulse is scattered by an obstacle, this method can be applied behind a reflected front, but not behind a diffracted one. To distinguish between these two types of front, consider for simplicity a convex obstacle which is fixed and rigid, and an incident pulse with an expanding front. The origin of time can be chosen at an instant when the obstacle is in the region not yet disturbed by the incident pulse. The surface of the obstacle can be divided into two domains: the reflecting region, every point of which can be joined to the initial position of the incident front by an incident ray in the exterior of the obstacles, and the shadow. The reflected fronts are the surfaces normal to the reflected rays, which issue from the reflecting region on the obstacle according to the reflection law of geometrical optics. In the shadow, the disturbance propagates over the obstacle with the speed of sound and is generally bounded by diffracted fronts which are involutes of the surface of the obstacle [2]. The expansion of the pressure in ascending powers of the time counted from the arrival of the front fails at a diffracted front because the surface of the obstacle, where the boundary conditions must be applied, is a caustic of the system of diffracted fronts.

It is therefore of some interest to investigate the behavior of the pressure behind a diffracted front in a specific case. The diffraction problems with known solutions - such as the half-plane and wedge problems - are instances where diffraction is essentially an edge effect. The simplest problem involving a convex obstacle is that of the diffraction by a circular cylinder of two-dimensional incident pulses in which the pressure is the same at all points of any straight line parallel to the axis of the cylinder. This is the problem discussed in this paper, attention being concentrated on the diffracted pressure field[*].

The problem which is here formulated and solved in terms of sound pulses can also be interpreted in electromagnetic terms. Suppose that the cylinder is a perfect conductor, and that the incident field is such that the only non-vanishing component of the magnetic field is that parallel to the axis of the cylinder. Then it satisfies the same boundary conditions, (1.6), at the cylinder, as the pressure does in the acoustic case. The other two-dimensional electromagnetic problem, in which the electric field is parallel to the axis of the cylinder, can be solved by the same technique; the modifications required are trival, and some of the calculations are actually simplified.

1. The Green's Function

To simplify the work, the radius of the cylinder will be taken as the unit of length, and the velocity of sound as the unit of velocity. Then the unit of time is the time taken by a pulse front to travel a distance equal to the radius. Only two-dimensional problems will be considered. With the axis of the cylinder as the z axis, the state of the medium is completely specified by giving the excess pressure p and the particle velocity components u, v, as functions of x, y and the time t. The equations of sound propagation are

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[*] A related problem is that of the propagation of a sound pulse in a stratified atmosphere bounded by a rigid plane, in which the sound velocity decreases with distance from the bounding plane. The case in which the reciprocal of the sound velocity increases linearly with the height has been investigated, and an account of it will, it is hoped, be published as a separate paper, by the author.

$$K \frac{\partial u}{\partial t} = - \frac{\partial p}{\partial x}, \quad K \frac{\partial v}{\partial t} = - \frac{\partial p}{\partial y}, \tag{1.1}$$

$$\frac{\partial p}{\partial t} + K\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = q \tag{1.2}$$

where K is the acoustic impedance (the product of density and sound velocity) of the medium, and a "source" term q has been included in the equation of continuity (1.2) as this is sometimes a convenient way of giving the incident pulse. A velocity potential ϕ can now be defined by the equations

$$p = \frac{\partial \phi}{\partial t}, \quad Ku = - \frac{\partial \phi}{\partial x}, \quad Kr = - \frac{\partial \phi}{\partial y}. \tag{1.3}$$

It satisfies the wave equation

$$\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = q. \tag{1.4}$$

The pressure p also satisfies the wave equation, with a forcing term $\frac{\partial q}{\partial t}$, and one can work with either ϕ or p.

If (r,θ) denote plane polar coordinates defined as usual by

$$x = r \cos \theta, \quad y = r \sin \theta \tag{1.5}$$

then the boundary condition at the cylinder, which expresses the vanishing of the normal component of the particle velocity, is

$$(1.6) \qquad \left[\frac{\partial \phi}{\partial r}\right]_{r=1} = \left[\frac{\partial}{\partial r}\int_{-\infty}^{t} p\,dt\right]_{r=1} = 0 \ .$$

An incident pulse may be specified in various ways. It may be given explicitly; if the origin of time is taken at an instant prior to that when the incident front touches the cylinder for the first time, this is equivalent to giving ϕ and $\frac{\partial \phi}{\partial t}$ when $t = 0$. Alternatively, these initial values may be given; they are equivalent to a distribution of instantaneous sources and doublets on the plane $t = 0$ in space time. Finally, a source distribution q which generates the incident pulse may be specified directly. In all cases, the diffraction problem may be solved if its Green's function is known. This may be defined as the solution of

$$(1.7) \qquad \frac{\partial^2 G}{\partial t^2} - \nabla^2 G = 2\pi\, \delta(x - x_o)\delta(y - y_o)\delta(t - t_o)$$

which satisfies the boundary condition

$$(1.8) \qquad \left[\frac{\partial G}{\partial r}\right]_{r=1} = 0$$

(here, as elsewhere, δ denotes the Dirac δ-function). The general case can obviously be deduced by rotation and translation from the special case $x = r_o$, $y_o = t_o = 0$, so that

$$(1.9) \qquad \frac{\partial^2 G}{\partial t^2} - \nabla^2 G = 2\pi\, \delta(r - r_o)\delta(y)\delta(t)$$

and this alone will be discussed here. If the cylinder were absent, the solution of (1.9) would be Hadamard's elementary solution

$$(1.10) \qquad G_o(x,y,z;r_o) = (t^2 - R^2)^{-1/2}\, H(t - R)$$

where H is Heaviside's unit function and

$$R = \left\{(x - r_o)^2 + y^2\right\}^{1/2} . \tag{1.11}$$

This is the incident field which generates the Green's function. (It must be understood throughout that $r_o \geq 1$).

It is natural to introduce the polar coordinates (1.5). One can then think of G as defined not only in the "physical plane" $(r \geq 0, -\pi \leq \theta \leq \pi)$ but also on a Riemann surface whose sheets are given by $(2m - 1)\pi \leq \theta \leq (2m + 1)\pi$, $m = \ldots, -1, 0, 1, \ldots$. Obviously, the instantaneous source at $x = r_o$, $y = 0$ then corresponds to an infinity of sources situated at the points $r = r_o$, $\theta = 2m\pi$ $(m = \ldots -1, 0, 1, \ldots)$. Thus (1.9) must be replaced by

$$\frac{\partial^2 G}{\partial t^2} - \nabla^2 G = \frac{2\pi}{r_o} \delta(r - r_o)\delta(t) \sum_{m=-\infty}^{\infty} \delta(\theta + 2m\pi) . \tag{1.12}$$

(The factor $1/r_o$ is required because the element of area of polar coordinates is $r dr d\theta$). If follows at once that one may put

$$G(r,\theta,t;r_o) = \sum_{m=-\infty}^{\infty} F(r,\theta + 2m\pi, t; r_o) \tag{1.13}$$

where F is defined for $-\infty < \theta < \infty$ by the equation

$$\frac{\partial^2 F}{\partial t^2} - \nabla^2 F = \frac{2\pi}{r_o} \delta(r - r_o)\delta(\theta)\delta(t) \tag{1.14}$$

and the boundary condition

$$\left[\frac{\partial F}{\partial r}\right]_{r = 1} = 0 . \tag{1.15}$$

One may think of F as the field due to an incident pulse generated by a modified "source" which has the requisite singularity only in the sheet $|\theta| \leq \pi$ of the Riemann surface. The formula (1.13) is thenan instance of the general reciprocity theorem.

2. The Laplace Transform of the Green's Function

Let

$$(2.1) \qquad \bar{F}(r,\theta,s;r_o) = \int_0^\infty F(r,\theta,t)e^{-st}\,dt$$

be the Laplace transform of F, and

$$(2.2) \qquad F^*(r,\nu,s;r_o) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \bar{F}(r,\theta,s;r_o)e^{-i\nu\theta}d\theta$$

be the Fourier transform of $\bar{F}$. Then[†]

$$(2.3) \qquad s^2\bar{F} - \nabla^2\bar{F} = \frac{2\pi}{r_o}\,\delta(r - r_o)\delta(\theta)$$

and

$$(2.4) \qquad \frac{d^2F^*}{dr^2} + \frac{1}{r}\frac{dF^*}{dr} - \left(\frac{\nu^2}{r^2} + s^2\right)F^* = -\frac{(2\pi)^{1/2}}{r_o}\,\delta(r - r_o)\ .$$

Also, it follows from (1.15) that

$$(2.5) \qquad \left[\frac{dF^*}{dr}\right]_{r=1} = 0\ .$$

[†] Since it is known that F is singular on certain characteristics in space-time, the statement that (2.3) is the Laplace transform of (1.13) is open to doubt. One can prove its validity by using the concept of the "finite part" of a divergent integral due to Hadamard, or by interpreting F as a "distribution" in the sense of L. Schwartz.

It may be assumed that F^* is continuous at $r = r_o$. Then (2.4) is equivalent to

$$(2.6) \qquad \frac{d^2F^*}{dr^2} + \frac{1}{r}\frac{dF^*}{dr} - \left(\frac{\nu^2}{r^2} + s^2\right) F^* = 0, \quad (r \neq r_o) ,$$

and

$$(2.7) \qquad \left[F^*\right]_{r_o - 0}^{r_o + 0} = 0, \quad \left[\frac{dF^*}{dr}\right]_{r_o - 0}^{r_o + 0} = -\frac{(2\pi)^{1/2}}{r_o}$$

Adding the condition $F^* \to 0$ as $r \to \infty$, one finds after a short calculation that

$$(2.8) \quad F^*(r,\nu,s;r_o) = \begin{cases} (2\pi)^{1/2} \dfrac{K_\nu(sr_o)}{K_\nu'(s)} \left\{ I_\nu(sr)K_\nu'(s) - I_\nu'(s)K_\nu(sr) \right\}, & (r \leq r_o) , \\[2ex] (2\pi)^{1/2} \dfrac{K_\nu(sr)}{K_\nu'(s)} \left\{ I_\nu(sr_o)K_\nu'(s) - I_\nu'(s)K_\nu(sr_o) \right\}, & (r \geq r_o) , \end{cases}$$

where I_ν, K_ν denote the modified Bessel functions of order ν of the first and second kind respectively. That (2.8) satisfies (2.5) is obvious; the validity of (2.7) follows from the identity

$$(2.9) \qquad I_\nu'(z)K_\nu(z) - I_\nu(z)K_\nu'(z) = \frac{1}{z} .$$

The Laplace transform $\overline{F}$ can now be obtained at once in the form

$$(2.10) \quad \overline{F}(r,\theta,s;r_o) = \int_{-\infty}^{\infty} \frac{I_\nu(sr)K_\nu'(s) - I_\nu'(s)K_\nu(sr)}{K_\nu'(s)} K_\nu(sr_o)e^{i\nu\theta}\, d\nu ,$$

for $r \leq r_o$; if $r \geq r_o$, then r and r_o must be interchanged.

It may be added that the Laplace transform of the Green's function,

$$(2.11) \qquad \overline{G}(r,\theta,s;r_o) = \sum_{m=-\infty}^{\infty} \overline{F}(r,\theta + 2m\pi, s;r_o) ,$$

can be converted into a Fourier series by using Poisson's summation formula. The relevant form of this is

$$(2.12)\qquad \sum_{m=-\infty}^{\infty} \overline{F}(r,\theta+2m\pi,s;r_o) = \frac{1}{(2\pi)^{1/2}} \sum_{n=-\infty}^{\infty} F^*(r,n,s;r_o)e^{in\theta} \quad ;$$

it can be obtained formally by calculating the Fourier coefficients of the left hand side[†]. Thus e.g. for $r \leq r_o$,

$$(2.13)\qquad \overline{G}(r,\theta,s;r_o) = \sum_{n=-\infty}^{\infty} \left\{K_n(sr_o)I_n(sr) - \frac{I_n'(s)K_n(sr_o)K_n(sr)}{K_n'(s)}\right\} e^{in\theta}$$

which can be written as

$$(2.14)\quad \overline{G}(r,\theta,s;r_o) = K_o(sR) - \frac{I_o'(s)K_o(sr_o)K_o(sr)}{K_o'(s)} - 2\sum_{n=1}^{\infty} \frac{I_n'(s)K_n(sr_o)K_n(sr)}{K_n'(s)} \cos n\theta ,$$

by the Addition Theorem for Bessel functions of imaginary argument; $K_o(sR)$ is the Laplace transform of the elementary solution (1.10). Conversely, (2.13) and (2.14) could have been obtained first and, using Poisson's summation formula, converted into (2.11) and (2.10). This method is legitimate but less transparent than that based on (1.12) (which is actually equivalent to (2.12)).

The integral representation (2.10) of a typical term of the series (2.11) is similar to the type of solution obtained in the discussion of the propagation of electromagnetic waves in a stratified atmosphere. (In fact, the problem of finding $\overline{F}$ is equivalent to such a case, as can be seen by making the substitution $r = \log z$). In the theory of propagation, it is found that the disturbance in the shadow can be obtained by evaluating the integrals in question by the method of residues in terms of a series of "propagation modes" (see for instance Kerr [4]). The same device can be applied to (2.10). For the integrand is, for fixed $s \neq 0$, a single-valued function of ν, considered as a complex variable. It can be shown, for s real and positive, that the contour can be moved across the upper or lower half-plane of ν according as $\theta \gtrless 0$; hence (2.10) can be replaced by the sum of the residues of its integrand at the poles in the upper or lower half-plane respectively. These are given by the zeroes of $K_\nu'(s)$ considered as a function of ν. It is shown in Appendix A that for s real and positive these zeroes are pure imaginary; hence they may be denoted by

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[†] Sufficient conditions for the validity of Poisson's formula, in a different form, but equivalent to (2.12), are given by E.C. Titchmarsh, Fourier Integrals, (Oxford 1937)p. 60.

$$(2.15) \qquad \nu = i\mu_j \qquad (j = 1,2,\ldots)$$

in the upper half-plane. As $K_\nu'(s)$ is even in ν, those in the lower half-plane are then $\nu = -i\mu_j$. Now

$$\left\{ I_\nu(sr)K_\nu'(s) - I_\nu'(s)K_\nu(sr) \right\} K_\nu(sr_o)$$

is also an even function of j ; thus the series of residues equivalent to (2.10) is formally

$$(2.16) \qquad \bar{F}(r,\theta,s;r_o) = 2\pi \sum_{j=1}^{\infty} \frac{I_{i\mu_j}'(s)K_{i\mu_j}(sr_o)K_{i\mu_j}(sr)}{\left[\frac{\partial}{\partial\mu} K_{i\mu}'(s)\right]_{\mu = \mu_j}} \exp(-\mu_j|\theta|)$$

and this is valid for both $r \leq r_o$ and $r \geq r_o$. Each term of (2.16) satisfies the Laplace transform of the wave equation, and is the Laplace transform of a " propagation mode" .

3. Approximation Valid for Large s

When s is large, (2.16) can be evaluated approximately. Since the behavior of $\bar{F}$ for large s determines the "initial" behavior of F, these approximations are of obvious interest in the theory of pulse diffraction. By hypothesis,

$\bar{F}$ is the Laplace transform of a function which vanishes for $t < 0$, and so is regular in some half-plane $\mathrm{Re}(s) > c > 0$. Hence it is sufficient to take s as real and positive in the first instance. Then various approximations can be used, which are derived in Appendix A.

In the first place, if $s \to \infty$ and the order j is fixed, then

$$\mu_j \sim s + \alpha_j s^{1/3} \tag{3.1}$$

where α_j is the j^{th} zero of $f'(\alpha)$, $f(\alpha)$ being defined by

$$f(\alpha) = \alpha^{1/2} \left\{ J_{1/3}\left(\frac{2^{3/2}}{3} \alpha^{3/2}\right) + J_{-1/3}\left(\frac{2^{3/2}}{3} \alpha^{3/2}\right) \right\} \tag{3.2}$$

This can also be written in terms of the Airy function[7],

$$\mathrm{Ai}(z) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{1}{3}\gamma^3 + \gamma z\right) d\gamma = \frac{z^{1/2}}{3^{1/2}\pi} K_{1/3}\left(\frac{2}{3} z^{3/2}\right), \tag{3.3}$$

as

$$f(\alpha) = \frac{3}{2^{1/6}} \mathrm{Ai}(-2^{1/3} \alpha) \quad . \tag{3.4}$$

Again, if from now on μ_j is understood to be given by (3.1) then

$$K_{i\mu_j}(s) \sim \frac{2^{1/2}\pi f(\alpha_j)}{3 s^{1/3}} \exp\left(-\frac{1}{2}\pi\mu_j\right) \quad , \tag{3.5}$$

$$\left[\frac{\partial}{\partial\mu} K'_{i\mu}(s)\right]_{\mu=\mu_j} \sim \frac{2^{3/2}\pi \alpha_j f(\alpha_j)}{3s} \exp\left(-\frac{1}{2}\pi\mu_j\right) \quad . \tag{3.6}$$

An approximation for $K_{i\mu_j}(sr)$ valid for $r \geq 1$ is obtained in Appendix A, but when it is substituted in (2.16) the result is too complicated for the calculation of the inverse Laplace transform. One is therefore forced to use a simpler formula, valid for $r > 1$ as $s \to \infty$ which is

$$(3.7) \qquad K_{i\mu_j}(sr) \sim \frac{\pi^{1/2}}{(2s)^{1/2}(r^2-1)^{1/4}} \exp\left\{-\frac{1}{2}\pi\mu_j - s\left[(r^2-1)^{1/2} - \cos^{-1}\frac{1}{r}\right] + a_j s^{1/3} \cos^{-1}\frac{1}{r}\right\}$$

where $\cos^{-1}(1/r)$ is defined to be between 0 and $\frac{1}{2}\pi$. For $r = 1$, (3.5) must be used. Also, it follows from (2.9) and the definition of μ_j that

$$I'_{i\mu_j}(s)\, K_{i\mu_j}(s) = \frac{1}{s}$$

whence, by (3.5)

$$(3.8) \qquad I'_{i\mu_j}(s) \sim \frac{3}{2^{1/2}\pi\, s^{2/3} f(a_j)} \exp\left(\frac{1}{2}\pi\mu_j\right) .$$

When substituting these approximations in (2.16), three cases must be distinguished, namely $r > 1$, $r_o > 1$; $r = 1$, $r_o > 1$ or $r > 1$, $r_o = 1$; and finally $r = r_o = 1$. In the first case, a typical term of (2.16) becomes

$$(3.9) \qquad \frac{9}{4(r_o^2-1)^{1/4}(r^2-1)^{1/4}} \qquad \frac{1}{a_j\left[f(a_j)\right]^2 s^{2/3}} \exp(-s\tau - a_j \delta s^{1/3})$$

where

$$(3.10) \qquad \delta = |\theta| - \cos^{-1}\frac{1}{r} - \cos^{-1}\frac{1}{r_o} ,$$

$$(3.11) \qquad \tau = \delta + (r^2-1)^{1/2} + (r_o^2-1)^{1/2} .$$

For $r = 1$, $r_o > 1$, one has

$$(3.12) \qquad \frac{3\pi^{1/2}}{2(r^2-1)^{1/4}} \qquad \frac{1}{a_j f(a_j) s^{1/2}} \exp(-s\tau - a_j\, \delta\, s^{1/3})$$

where δ and τ must now be evaluated with $r = 1$. The typical term for $r = 1$, $r > 1$ is obtained from (3.12) by replacing r_o by r. Finally, if $r = r_o = 1$ then

$$\tau = \delta = |\theta| \quad (r = r_o = 1) \tag{3.13}$$

and a typical term is

$$\frac{1}{a_j s^{1/3}} \exp\left\{- |\theta|(s + a_j s^{1/3})\right\} . \tag{3.14}$$

The fractional errors of these approximations are all, for fixed r and r_o, of order $s^{-1/3}$.

For large j one obtains by substituting the asymptotic formulae for the Bessel functions in (3.2) that

$$f(a) \sim \frac{3}{\pi^{1/2}(2a)^{1/4}} \cos\left(\frac{2^{3/2}a^{3/2}}{3} - \frac{\pi}{4}\right) . \tag{3.15}$$

Hence, as $j \to \infty$,

$$a_j \sim \frac{1}{2}(3\pi)^{2/3}\left(j - \frac{3}{4}\right)^{2/3} . \tag{3.16}$$

Thus the series (3.9), (3.12) and (3.14) converge only if $\delta > 0$. In the analogous problem of propagation in a stratified atmosphere, which has already been referred to, one can show that in the shadow the asymptotic behavior of the sum of the series corresponding to that for F can be inferred from the behavior of the individual terms. Outside the shadow this is not the case, although the series converges. It seems reasonable to assume that the same result is valid in the problem treated here.

4. Approximate Formulae for the Green's Function

The evaluation of the inverse Laplace Transform of (3.9), (3.12) and (3.14), with $j = 1$, involves the calculation of the integral

$$U_k(\alpha_1\delta,T) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} s^{-k}\exp(sT - \alpha_1\delta\, s^{1/3})ds \tag{4.1}$$

where

$$T = t - \tau \quad , \tag{4.2}$$

$k = \frac{2}{3}, \frac{1}{3}$ or $\frac{1}{2}$, and $c > 0$. This integral will also be required later for other values of k. If $T < 0$, the contour can be deformed into a large semi-circle in $\mathrm{Re}(s) > 0$. The integral taken along this contour vanishes in the limit, by Jordan's Lemma. Hence U_k vanishes for $T < 0$.

For $T \geq 0$, U_k can be estimated by the method of steepest descents. The function $sT - \alpha_1\delta^{1/3}$ of s has a saddle point when

$$s = s_o = (\alpha_1\delta/3T)^{3/2}$$

and it is easily shown that

$$sT - \alpha_1\delta\, s^{1/3} = -\frac{2(\alpha_1\delta)^{3/2}}{3^{3/2}\,T^{1/2}} + \frac{3^{1/2}\,T^{5/2}}{(\alpha_1\delta)^{3/2}}(s-s_o)^2 + \dots \quad .$$

Hence on the path of steepest descent through the saddle point

$$s = s_o + i\,\frac{(\alpha_1\delta)^{3/4}}{3^{1/4}\,T^{5/4}}\,\zeta + \dots \quad ,$$

with real ζ, and so

$$U_k(\alpha_1\delta,T) \sim \frac{3^{(6k-1)/4}}{2\pi^{1/2}}(\alpha_1\delta)^{(3-6k)/4}\,T^{(6k-5)/4}\,H(T)\exp(-\xi_1) \tag{4.3}$$

where

$$\xi_1 = \frac{2(\alpha_1\delta)^{3/2}}{3^{3/2}\,T^{1/2}} \quad , \tag{4.4}$$

and the factor $H(T)$ has been inserted because $U_k = 0$ for $T < 0$. Using (4.3), one finds the following approximations for F from (3.9), (3.12) and (3.15):

$$(4.5)\qquad F(r,\theta,t;r_0) \sim \frac{3^{11/4}\, H(T)}{8\pi^{1/2}\left\{(r_0^2-1)(r^2-1)T\delta\right\}^{1/4}}\;\frac{\exp(-\xi_1)}{\alpha_1^{5/4}\left[f(\alpha_1)\right]^2}\,,$$

$$(4.6)\qquad F(1,\theta,t;r_0) \sim \frac{3^{3/2}\, H(T)}{4(r_0^2-1)^{1/4}\, T^{1/2}}\;\frac{\exp(-\xi_1)}{\alpha_1 f(\alpha_1)}\,,$$

$$(4.7)\qquad F(1,\theta,t;1) \sim \frac{3^{1/4}\,\pi^{1/2}|\theta|^{1/4}\, H(T)}{2T^{3/4}}\;\frac{\exp(-\xi_1)}{\alpha_1^{3/4}}\,,$$

where, in (4.7), $T = t - |\theta|$ and $\delta = |\theta|$, in accordance with (3.14).

If $3k$ is an integer then U_k can be evaluated explicitly. Take first $k = \frac{2}{3}$ and put $sT = \frac{1}{3} s_1^3$ in (4.1). Then

$$U_{2/3}(\alpha_1\delta,T) = \frac{3^{2/3}\, T^{-1/3}}{2\pi i}\int_{\infty\exp(-i\pi/6)}^{\infty\exp(i\pi/6)} \exp\left\{\frac{1}{3}s_1^3 - \alpha_1\delta(3T)^{-1/3}\, s_1\right\} ds_1\,.$$

Since $\mathrm{Re}(s_1^3) \leq 0$ for $\frac{1}{6}\pi \leq |\arg s_1| \leq \frac{1}{2}\pi$, the contour can be deformed into the imaginary axis, and so

$$U_{2/3}(\alpha_1\delta,T) = \frac{3^{2/3}\, T^{-1/3}}{2\pi}\int_{-\infty}^{\infty} \exp\left\{-\, i\left[\frac{1}{3}s_2^3 + \alpha_1\delta(3T)^{-1/3}\, s_2\right]\right\} ds_2\,.$$

Thus according to the definition (3.3) of the Airy function,

$$(4.8)\quad U_{2/3}(\alpha_1\delta,T) = 3^{2/3}\, T^{-1/3}\, H(T)\mathrm{Ai}\left\{\alpha_1\delta(3T)^{-1/3}\right\} = \frac{1}{\pi}\left(\frac{\alpha_1\delta}{T}\right)^{1/2} H(T)K_{1/3}(\xi_1)$$

where ξ_1 is again given by (4.4).

The value of U_k when $3k$ is an integer can be deduced by repeated differentiation or integration with respect to the parameter $\alpha_1\delta$ from (4.8) since clearly

$$\frac{\partial U_k(\alpha_1\delta, T)}{\partial(\alpha_1\delta)} = -U_{(3k-1)/3}(\alpha_1\delta, T) \quad . \tag{4.9}$$

Hence one can use, instead of (4.5) and (4.7), the formulae

$$\left\{\begin{aligned} F(r,\theta,t;r_o) &\sim \frac{3^{8/3}\, H(T)}{4(r_o^2-1)^{1/4}(r^2-1)^{1/4}\, T^{1/3}} \; \frac{\mathrm{Ai}\left\{\alpha_1\delta(3T)^{-1/3}\right\}}{\alpha_1\left[f(\alpha_1)\right]^2} = \\ &= \frac{9\delta^{1/2}\, H(T)}{4\,\pi(r_o^2-1)^{1/4}(r^2-1)^{1/4}\, T^{1/2}} \; \frac{K_{1/3}(\xi_1)}{\alpha_1^{1/2}\left[f(\alpha_1)\right]^2} \quad , \end{aligned}\right. \tag{4.10}$$

$$\left\{\begin{aligned} F(1,\theta,t;1) &\sim -\frac{3^{1/3}\pi\, H(T)}{T^{2/3}} \; \frac{1}{\alpha_1} \; \mathrm{Ai}'\left\{\alpha_1\delta(3T)^{-1/3}\right\} = \\ &= -H(T)\left\{\frac{K_{1/3}(\xi_1)}{2\,\alpha_1^{3/2}|\theta|^{1/2}\, T^{1/2}} + \frac{|\theta|K'_{1/3}(\xi_1)}{3^{1/2}\, T}\right\} \quad . \end{aligned}\right. \tag{4.11}$$

The fractional error of these approximations is of order $T^{1/3}$. Finally, the corresponding approximations for the Green's function are obtained by substituting the appropriate form of F in (1.13).

5. Interpretation in Terms of Geometrical Optics

To interpret these results it is best to begin with the typical constituent pulse F. This, it will be recalled, is the field due to an incident pulse defined on a Riemann surface whose sheets are $(2m-1)\pi \leq \theta \leq (2m+1)\pi$ due to a singularity which has the character of an instantaneous source at time $t = 0$, at $r = r_o$, $\theta = 0$. If the cylinder were absent, the fronts of this pulse would be the circles

$$t = R = \left(r^2 + r_o^2 - 2rr_o\cos\theta\right)^{1/2} \tag{5.1}$$

at least for $0 < t \leq r_o$. The fronts of the secondary disturbance can be deduced from Fermat's principle, since the arguments by which the validity of this principle is established for pulse fronts[2] can be extended to the case of propagation of a Riemann surface. There is a family of reflected fronts E, and the two families of diffracted fronts D', D''. The fronts E are the normal trajectories of the rays obtained by reflecting the incident rays - which are simply the straight lines through $r = r_o$, $\theta = 0$ - at their intersections with $r = 1$ nearest to the source. These rays issue from the arc $|\theta| \leq \cos^{-1}\frac{1}{r_o}$, and the fronts R remain in the sheet $m = 0$. For $0 < t \leq r_o - 1$, there is no secondary disturbance at all; for $r_o - 1 \leq t \leq \left(r_o^2 - 1\right)^{1/2}$, only a reflected pulse is present; but for $t \geq \left(r_o^2 - 1\right)^{1/2}$, diffraction into the shadow takes place. On the Riemann surface the shadow, which is bounded by the prolongation of the incident rays which touch $r = 1$, is given by the inequalities

$$\theta > \cos^{-1}\frac{1}{r} + \cos^{-1}\frac{1}{r_o}\,, \qquad \theta < -\cos^{-1}\frac{1}{r} - \cos^{-1}\frac{1}{r_o}\,, \tag{5.2}$$

or, by (3.10)

$$\delta > 0\,. \tag{5.3}$$

To determine the arrival time of the diffracted pulse, and the diffracted fronts D', D'', one may begin by considering the points on $r = 1$. One front, D', propagates in the sense of increasing θ, starting at $\theta = \cos^{-1}\frac{1}{r_o}$ when $t = \left(r_o^2-1\right)^{1/2}$. It propagates onto the sheets with $m > 0$. The other front, D'', starts at $\theta = -\cos^{-1}\frac{1}{r_o}$ when $t = \left(r_o^2-1\right)^{1/2}$ and encircles the cylinder in the direction of decreasing θ, propagating into the sheets with $m < 0$. Thus by (3.11) the equation of both fronts may be written as

$$t = \tau \tag{5.4}$$

so that τ simply denotes the arrival time of the diffracted front (only one diffracted front at most arrives at a point on the Riemann surface). The rays normal to D' and D'' are in general tangent to $r = 1$; for instance, the ray normal to D' through (r,θ) touches $r = 1$ at $\theta - \cos^{-1}\frac{1}{r_o}$, and is of length $(r^2 - 1)^{1/2}$. Hence, again by (3.11), (5.4) is also the joint equation of D' and D'' at all points of the shadow. Thus T is the time counted from the arrival of the diffracted front, and the factor $H(T)$ in all the approximations obtained in Section 4 implies that the disturbance starts when the diffracted front arrives; this agrees, as it should, with the general theory of diffracted pulses.

Now G is given by (1.13),

$$G(r,\theta,t;r_o) = \sum_{m=-\infty}^{\infty} F(r,\theta + 2m\pi,t;r_o) \quad . \tag{5.5}$$

Since G has period 2π with respect to θ, it is sufficient to take $0 \leq |\theta| \leq \pi$. Let

$$\begin{aligned} \delta_o &= |\theta| - \cos^{-1}\frac{1}{r} - \cos^{-1}\frac{1}{r_o} \quad , \\ \delta_m &= 2m\pi + \theta - \cos^{-1}\frac{1}{r} - \cos^{-1}\frac{1}{r_o} \quad , \qquad (m > 0) \, , \\ \delta_m &= - 2m\pi - \theta - \cos^{-1}\frac{1}{r} - \cos^{-1}\frac{1}{r_o} \quad , \qquad (m < 0) \, , \end{aligned} \tag{5.6}$$

and

$$\tau_m = \delta_m + (r^2 - 1)^{1/2} + (r_o^2 - 1)^{1/2} \quad , \qquad T_m = t - \tau_m \quad . \tag{5.7}$$

Consider first the case of a point (r,θ) in the shadow, so that $\delta_o > 0$. Then the term " m" of (5.5) is zero for $t \leq \tau_m$. Now the diffracted fronts in the physical plane are obtained by " projecting" D' and D'' from the sheets of the Riemann surface onto the sheet $m = 0$. Diffraction begins with $t = (r_o^2 - 1)^{1/2}$ at the two points

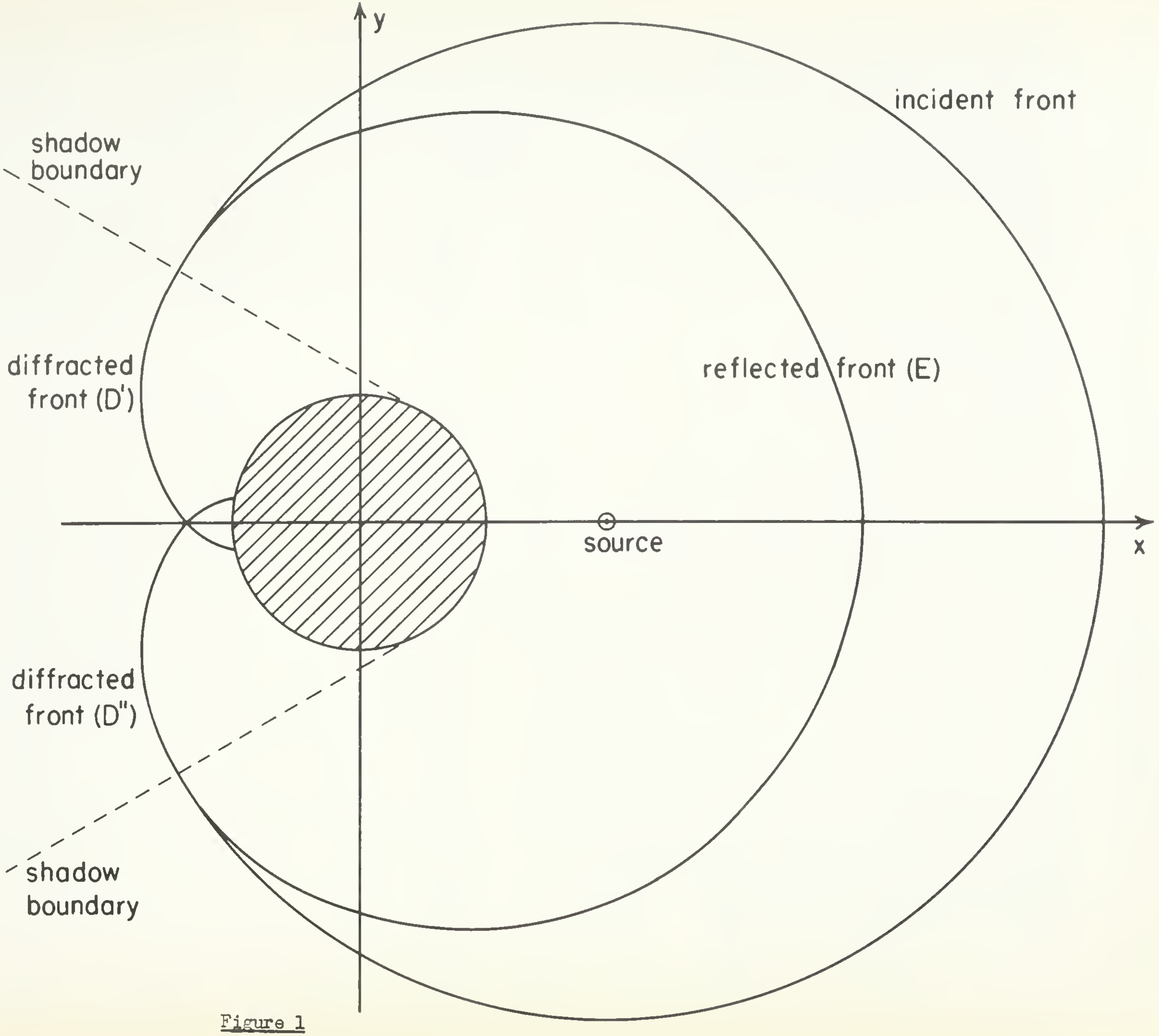
y
x
incident front
reflected front (E)
source
shadow boundary
diffracted front (D')
diffracted front (D")
shadow boundary

Figure 1

$r = 1$, $\theta = \pm \cos^{-1}(1/r_o)$, and D', D'', may be described as 'coming from' these points respectively. As t increases, the end-points of D' and D'' travel steadily round the cylinder in opposite directions. Thus the diffracted fronts meet at $r = 1$, $\theta = \pi$ when

$$t = \pi + (r_o^2 - 1)^{1/2} - \cos^{-1}\frac{1}{r_o} \,,$$

and subsequently penetrate into the region in which the secondary disturbance has already been established. These fronts, and the reflected ones, are shown in Fig. 1.

By symmetry, it is sufficient to consider the case $\theta > 0$. Then the first front to arrive is obviously D'. It arrives at time τ_o, and G is then given by $F(r,\theta,t;r_o)$. Subsequently, D' passes (r,θ) when $t = \tau_o + 2m\pi = \tau_m$, $(m = 1,2,\ldots)$. Each time that it does so, a further contribution $F(r,\theta + 2m\pi,t;r_o)$ must be added to G. Similarly, D'' passes (r,θ) when $t = \tau_m$, with $m = -1,-2,\ldots$ and the term $F(r,\theta - 2m\pi,t;r_o)$ must be added to G on each occasion. When $\theta < 0$, the parts played by D' and D'' are reversed; on $\theta = \pi$, both arrive simultaneously, and the terms with m and $-m-1$ in (5.5) are equal to each other.

It remains to discuss the case of a point (r,θ) in the reflection region $(\delta_o < 0)$. The evaluation of (2.10) by residues is then invalid for the term $m = 0$ of (5.5). The other terms still represent diffracted pulses. To deal with the term $m = 0$, one can apply the method of steepest descent to (2.10). It is found that on $r = 1$, $F(r,\theta,t;r_o)$ is simply equal to the doubled elementary solution (1.10) to order $(t-R)^{-1/2}$, which is to be expected. One may therefore identify $F(r,\theta,t;r_o)$ with the sum of the incident and the reflected pulse when $\delta_o < 0$. The calculations are rather involved, and do not furnish any new information; as they are however of some analytical interest, they are given in outline in Appendix B.

6. Numerical Results for the Case $r_o = r = 1$

It is clear that the approximations obtained, which are valid when T is small, can only be applied to the calculation of G when (5.5) reduces to the term $m = 0$, or, in the neighborhood of $\theta = \pi$, to the terms $m = 0$, $m = -1$. One can of course also estimate the additional contributions made by the other terms, provided that T_m is small.

Some calculations were made for the case $r_o = r = 1$, which is probably typical, using the value

$$\frac{2}{3^{3/2}} \alpha_1^{3/2} = 0.279 \quad . \tag{6.1}$$

Now it is apparent, from (4.7) and (4.12), that F and all its derivatives with respect to T vanish for $T = 0$. (In fact, F has an essential singularity at the diffracted front, and this is the reason why the diffracted pulse cannot be expanded in ascending powers of T). This may be contrasted with the behavior of the reflected pulse at its fronts where G becomes infinite like $T^{-1/2}$ if T is the time counted from the arrival of the reflected front. However, as T increases, $F(1,\theta,t;1)$ very soon increases; and as $\delta_o \to 0$, the initial rise becomes very rapid. This is illustrated by the curves plotted in Fig. 2, which show G as a function of T for $\theta = \pi$, $\theta = \frac{1}{2}\pi$ and $\theta = \frac{1}{4}\pi$. To bring these curves together into one figure, a different scale had to be employed for the ordinate in each case. Thus the curves are the graphs of $10\, G(1, \pi, t, 1)$, $0.5\, G(1, \frac{1}{2}, \pi, t; 1)$ and of $0.1\, G(1, \frac{1}{4}, \pi, t; 1)$. The abscissa is $T = t - \theta$.

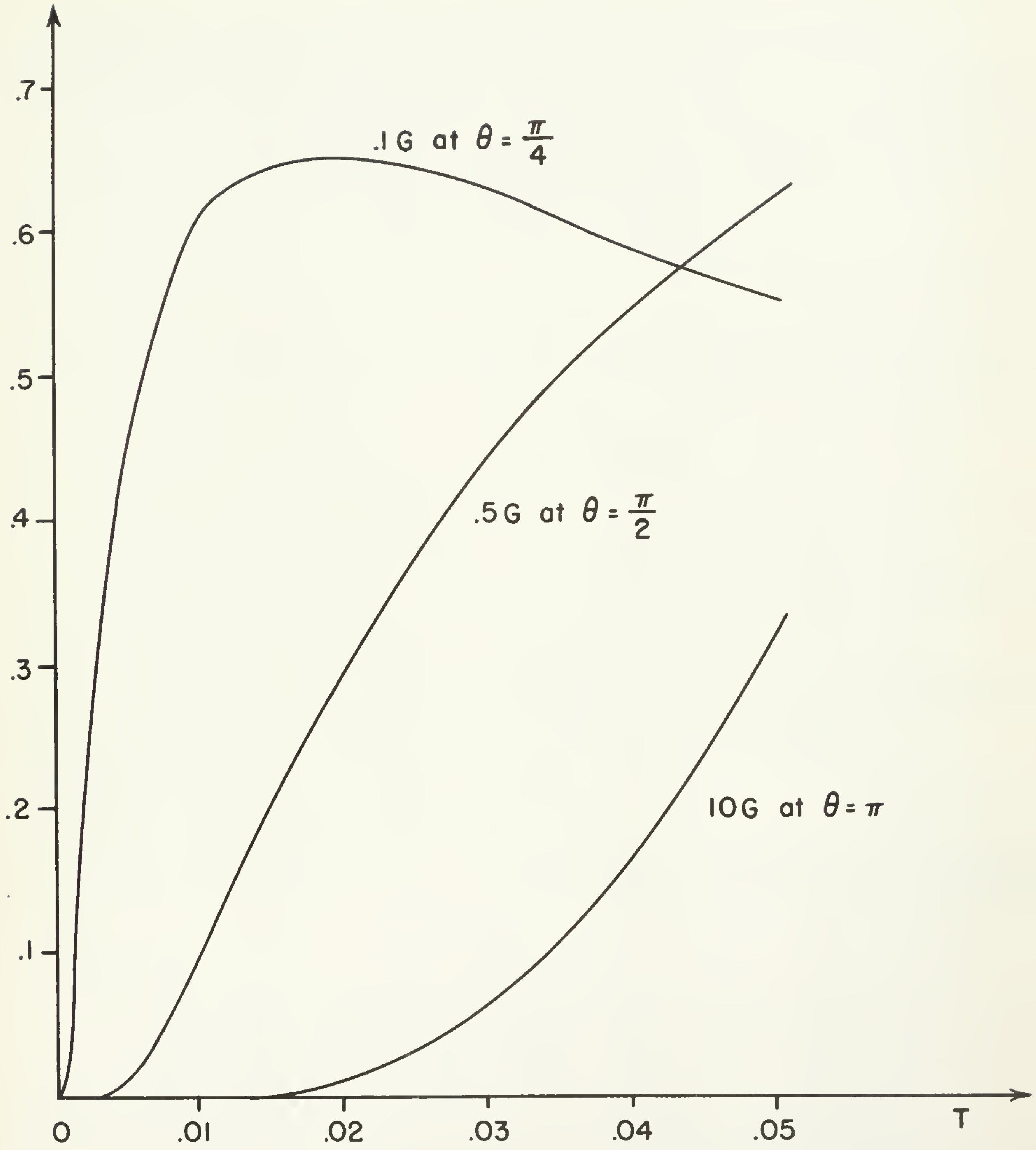
.7
.6
.5
.4
.3
.2
.1
0
.01
.02
.03
.04
.05
T
.1G at $\theta = \frac{\pi}{4}$
.5G at $\theta = \frac{\pi}{2}$
10G at $\theta = \pi$

Figure 2

The values of G at $\theta = \pi$ may be compared with those of the elementary solution G_o given by (1.10), which would be obtained if the cylinder were absent, when the times elapsed since the arrival of the respective fronts are equal. The front of the elementary solution arrives when $t = 2$, and with $T^* = t - 2$ one has

$$G_o = \left\{T^*(T^* + 4)\right\}^{-1/2} H(T^*) \quad . \tag{6.2}$$

The following table shows the values of G - calculated by means of (4.12) - and of G_o, for $0 \leq T \leq 0.4$ and $T = T^*$.

TABLE 1 ($\theta = \pi$)

$T = T^*$	0	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40
G	0	0.03	0.15	0.27	0.37	0.47	0.53	0.58	0.63
G_o	∞	2.22	1.56	1.27	1.09	0.97	0.88	0.81	0.75

Thus the cylinder becomes ineffective as a " shield" from the source for pulses whose length is equal to about half the radius.

For small θ, one may compare G with the solution obtained when the cylinder is replaced by the rigid plane $x = 1$. This is simply $2\,G_o$, and one may compare the value of G at θ with that of $2\,G_o$ evaluated at a distance θ from the source. This gives the following table at $\theta = \frac{1}{4}\pi$:

TABLE 2 ($\theta = \frac{1}{4}\pi$)

T	0	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40
G	0	5.50	4.30	3.65	3.29	3.01	2.84	2.72	2.61
$2G_o$	∞	7.02	4.90	3.94	3.37	2.97	2.67	2.45	2.26

7. Diffraction of Plane Pulses

The diffracted field corresponding to any incident pulse can be obtained by means of the Green's function. As an example, the diffraction of a plane pulse will be considered. Let the incident pulse be given by

$$(7.1) \qquad p = p_o(t + x) = p_o(t + r \cos \theta)$$

where p denotes the excess pressure. It is sufficient to take

$$(7.2) \qquad p_o(t + x) = \delta(t + x)$$

where δ denotes the Dirac δ-function. For if $P(r,\theta,t)$ denotes the corresponding diffracted field, then the diffracted field is given in general by

$$(7.3) \qquad p(r,\theta,t) = \int_{-\infty}^{\infty} P(r,\theta,t-t')p_o(t')dt' .$$

Now (7.2) is, formally, the inverse Laplace transform of $\exp(-sr \cos \theta)$. But the Laplace transform of the elementary solution (1.10) is

$$(7.4) \qquad \bar{G}_o(r,\theta,s;r_o) = K_o(sR), \quad R = (r^2 + r_o^2 - 2rr_o \cos \theta)^{1/2} ,$$

and so, for large r_o,

$$\bar{G}(r,\theta,s;r_o) = \left(\frac{\pi}{2sR}\right)^{1/2} e^{-sR}\left\{1 + O\left(\frac{1}{sR}\right)\right\} = \left(\frac{\pi}{2sr_o}\right)^{1/2} e^{-sr_o} e^{sr \cos \theta}\left\{1 + O\left(\frac{1}{r_o}\right)\right\} .$$

It follows that

$$(7.5) \qquad \lim_{r_o \to \infty} \left\{\left(\frac{2sr_o}{\pi}\right)^{1/2} e^{sr_o} \bar{G}_o(r,\theta,s)\right\} = e^{-sr \cos \theta}$$

and hence obviously also

$$(7.6) \qquad \lim_{r_o \to \infty} \left\{\left(\frac{2r_o s}{\pi}\right)^{1/2} e^{sr_o} \bar{G}(r,\theta,s)\right\} = \bar{P}(r,\theta,s) .$$

By (1.13) and (2.16) the diffracted field P is therefore given by

(7.7) $$P(r,\theta,t) = \sum_{m=-\infty}^{\infty} Q(r,\theta + 2m\pi,t)$$

where

(7.8) $$\bar{Q}(r,\theta,s) = 2\pi \sum_{j=1}^{\infty} \frac{I'_{i\mu_j}(s) K_{i\mu_j}(sr)}{\left[\frac{\partial}{\partial\mu} K'_{i\mu}(s)\right]_{\mu=\mu_j}} \exp(-\mu_j|\theta|) \quad .$$

When $|s|$ is large, $\bar{Q}$ can be approximated by means of the formulae (3.1) to (3.8); alternatively, the approximations (3.9) and (3.12) can be substituted for $\bar{F}$ before making $r_0 \to \infty$ in (7.6). It is found in this way that

(7.9) $$\bar{Q}(r,\theta,s) \sim \frac{9}{2^{3/2}\,\pi^{1/2}(r^2-1)^{1/4}} \quad \frac{1}{\alpha_1\left[f(\alpha_1)\right]^2 s^{1/6}} \exp(-s\delta - \alpha_1 \delta s^{1/3})$$

when $r > 1$, and

(7.10) $$\bar{Q}(1,\theta,s) \sim \frac{3}{\sqrt{2}} \quad \frac{1}{\alpha_1 f(\alpha_1)} \exp\left\{-(|\theta| - \tfrac{1}{2}\pi)(s + \alpha_1 s^{1/3})\right\}$$

where δ is now obtained from (3.10) by letting $r_0 \to \infty$,

(7.11) $$\delta = |\theta| - \frac{1}{2}\pi - \cos^{-1}\frac{1}{r}$$

and

(7.12) $$\sigma = \lim_{r_0 - \infty} (\tau + r_0) = \delta + (r^2 - 1)^{1/2} \quad .$$

For $r = 1$

(7.13) $$\delta = \sigma = |\theta| - \frac{1}{2}\pi \;(r = 1) \quad .$$

The series (7.9) and (7.10) converge only when $\delta > 0$. This again corresponds to a diffracted pulse, for the shadow boundaries are now $y = \pm 1$, $x < 0$ (they are in the sheet $|\theta| \leq \pi$ of the Riemann surface on which Q is defined).

When $\delta > 0$, then Q vanishes for $t \leq \sigma$; $t = \sigma$ is the equation of the two diffracted fronts on the Riemann surface corresponding to the incident front

$t = -x = -r \cos \theta$. Thus the contribution of the term $Q(r,\theta,t)$ of (7.7) to the Faltung integral (7.3) is

$$\int_{\sigma}^{t} p_o(t - t')Q(r,\theta,t')dt' \tag{7.14}$$

provided that $p_o(t) = 0$ for $t < 0$. Since the approximations for $Q(r,\theta,t)$ which can be deduced from (7.9) and (7.10) are valid only when $t - \sigma$ is small, (7.14) can be applied to " short" pulses only; in particular, P itself can be considered as giving the diffracted field of a short " square" pulse of length L, if it is multiplied by L.

8. Evaluation of the Diffracted Field

The inverse Laplace transforms of (7.9) and (7.10) are, in the notation of Section 4,

$$Q(r,\theta,t) \sim \frac{9}{2^{3/2}\, \pi^{1/2}(r^2 - 1)^{1/4}} \; \frac{U_{1/6}(\alpha_1\delta,T)}{\alpha_1\left[f(\alpha_1)\right]^2} , \tag{8.1}$$

$$Q(1,\theta,t) \sim \frac{3}{\sqrt{2}} \; \frac{U_o(\alpha_1\delta,T)}{\alpha_1 f(\alpha_1)} , \tag{8.2}$$

where now

$$T = t - \sigma , \tag{8.3}$$

so that T is again the time counted from the arrival of the diffracted pulse. According to (4.3), (8.1) and (8.2) give

$$Q(r,\theta,t) \sim \frac{9\delta^{1/2}\ H(T)}{2^{5/2}\, \pi(r^2 - 1)^{1/4}\ T} \; \frac{\exp(-\xi_1)}{\alpha_1^{1/2}\left[f(\alpha_1)\right]^2} , \tag{8.4}$$

$$Q(1,\theta,t) \sim \frac{3^{3/4}\ \delta^{3/4}\ H(T)}{2^{3/2}\ \pi^{1/2}\ T^{5/4}} \; \frac{\exp(-\xi_1)}{\alpha_1^{1/4}\ f(\alpha_1)} , \tag{8.5}$$

where again

$$\xi_1 = \frac{2}{3^{3/2}} \frac{(\alpha_1\delta)^{3/2}}{T^{1/2}} \quad . \tag{8.6}$$

Also, by (4.10) and (4.8),

$$\begin{cases} U_o(\alpha_1\delta,T) = \frac{H(T)}{T} \, Ai'' \left\{\alpha_1\delta(3T)^{-1/3}\right\} = \frac{\alpha_1\delta \, H(T)}{3^{1/3} \, T^{4/3}} \, Ai \left\{\alpha_1\delta(3T)^{-1/3}\right\} \\ \qquad\qquad = \frac{1}{3\pi}\left(\frac{\alpha_1\delta}{T}\right)^{3/2} H(T)K_{1/3}(\xi_1) \end{cases} \tag{8.7}$$

so that one may replace (8.5) by

$$Q(1,\theta,t) \sim \frac{\delta^{3/2} \; H(T)}{2^{1/2} \, \pi \, T^{3/2}} \, \frac{\alpha_1^{1/2}}{f(\alpha_1)} \, K_{1/3}(\xi_1) \quad . \tag{8.8}$$

As in the case of the Green's functions, the series (7.7) reduces to a finite sum. Each term is a contribution which only differs from zero after the passage of one of the diffracted fronts across (r,θ). In particular, if one takes $\frac{1}{2}\pi + \cos^{-1}\frac{1}{r} < \theta < \pi$, then the first pulse to arrive is $Q(r,\theta,t)$. Its front propagates in the positive sense round the cylinder, and at its m-th passage the term $Q(r,\theta + 2(m+1)\pi,t)$ must be added to P. The other set of pulses associated with a front propagating in the negative sense round the cylinder, is given by the terms of (7.7) with $m < 0$. At $\theta = \pi$, the terms with m and $-m-1 (m > 0)$ are equal to each other. The total excess pressure P can only be calculated by means of (8.4) and (8.5) or (8.8) immediately after the first arrival of the diffracted front; one can however say that in general the additional contributions due to the terms with $m \geq 1$ or $m \leq -2$ are negligibly small.

When the values of α_1, $f(\alpha_1)$ are substituted in (8.4),(8.5), these equations become

$$(8.9) \qquad Q(r,\theta,t) \sim 3.54 \frac{\xi_1^2 \exp(-\xi_1)}{(r^2-1)^{1/4}\,\delta^{5/2}} H(T) ,$$

$$(8.10) \qquad Q(1,\theta,t) \sim 8.16 \frac{\xi_1^{5/2} \exp(-\xi_1)}{\delta^3} H(T) ,$$

with

$$(8.11) \qquad \xi_1 = 0.28 \frac{\xi_1^{3/2}}{T^{1/2}} .$$

Some curves of P against T computed from these formulae are shown in Fig. 3 and 4. Figure 3 gives (P,T) curves for $r = 1$ and $\delta = \theta - \frac{1}{2}\pi$ equal to 30°, 45°, 60° and 90°; Fig. 4 gives the (P,T) curves on $\theta = \pi$ for the same values of δ, which correspond to $r = 2$, 1.414, 1.155 and 1.0. Both sets of curves show that for $T > 0.05$ P is roughly independent of δ and r; they also illustrate how the diffraction effect lessens as one moves away from the point $r = 1$, $\theta = \pi$ either along the cylinder or along the line $\theta = \pi$. At $r = 1$, $\theta = \pi$ the infinitely short incident pulse is converted into a pulse consisting of a roughly linear pressure rise followed by a very slow decrease, so that diffraction appears to have an integrating effect.

9. The Diffraction of Short Harmonic Waves

The diffraction of harmonic waves can be treated by the simple device of putting $s = i\omega$ in the Laplace transform, where $\lambda = 2\pi/\omega$ is the ratio of the wave-length of the incident waves to the radius of the cylinder, and multiplying by $e^{i\omega t}$. For instance, if the incident waves are plane, so that

$$(9.1) \qquad p_o(t) = e^{i\omega t} ,$$

then (7.3) gives

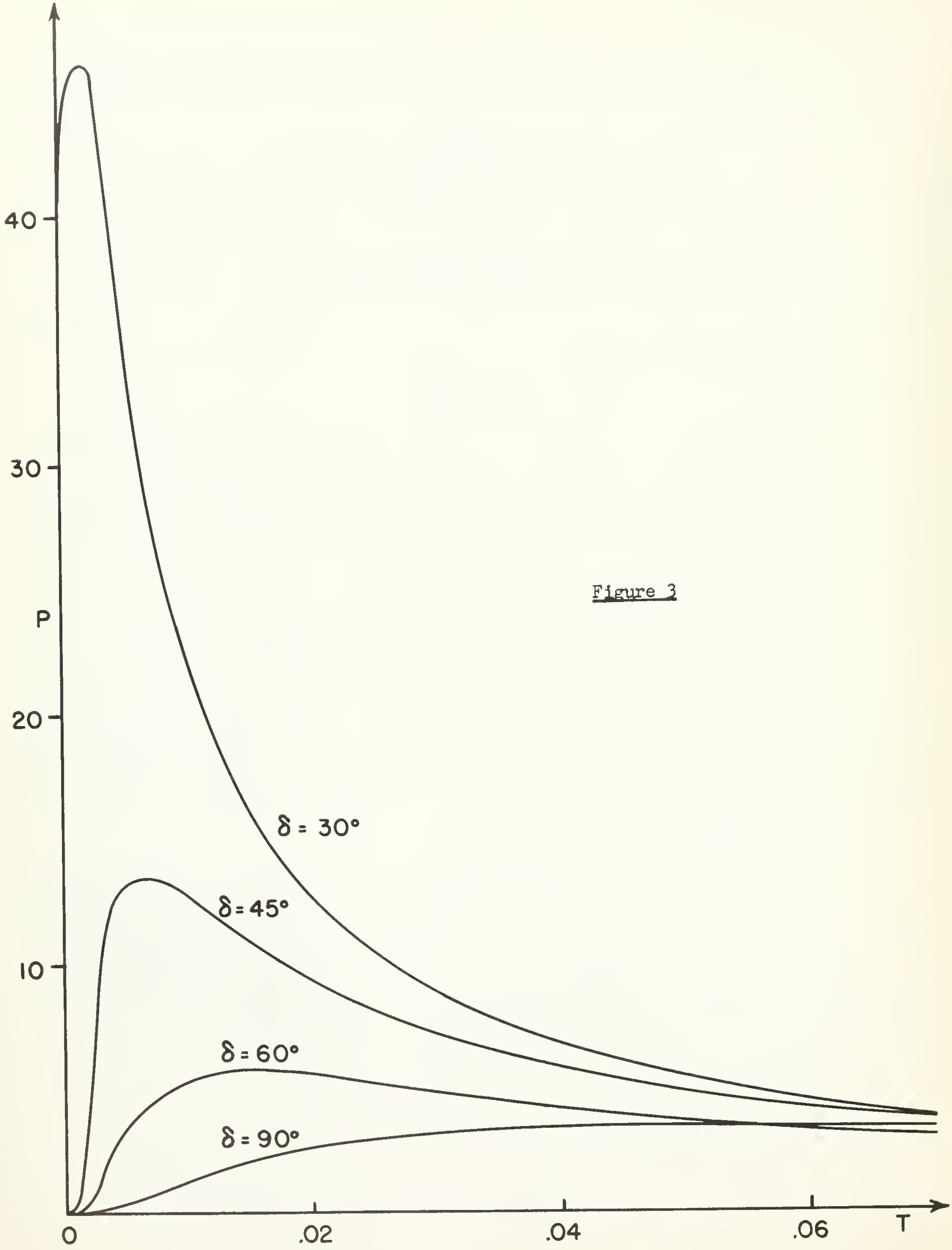

Figure 3

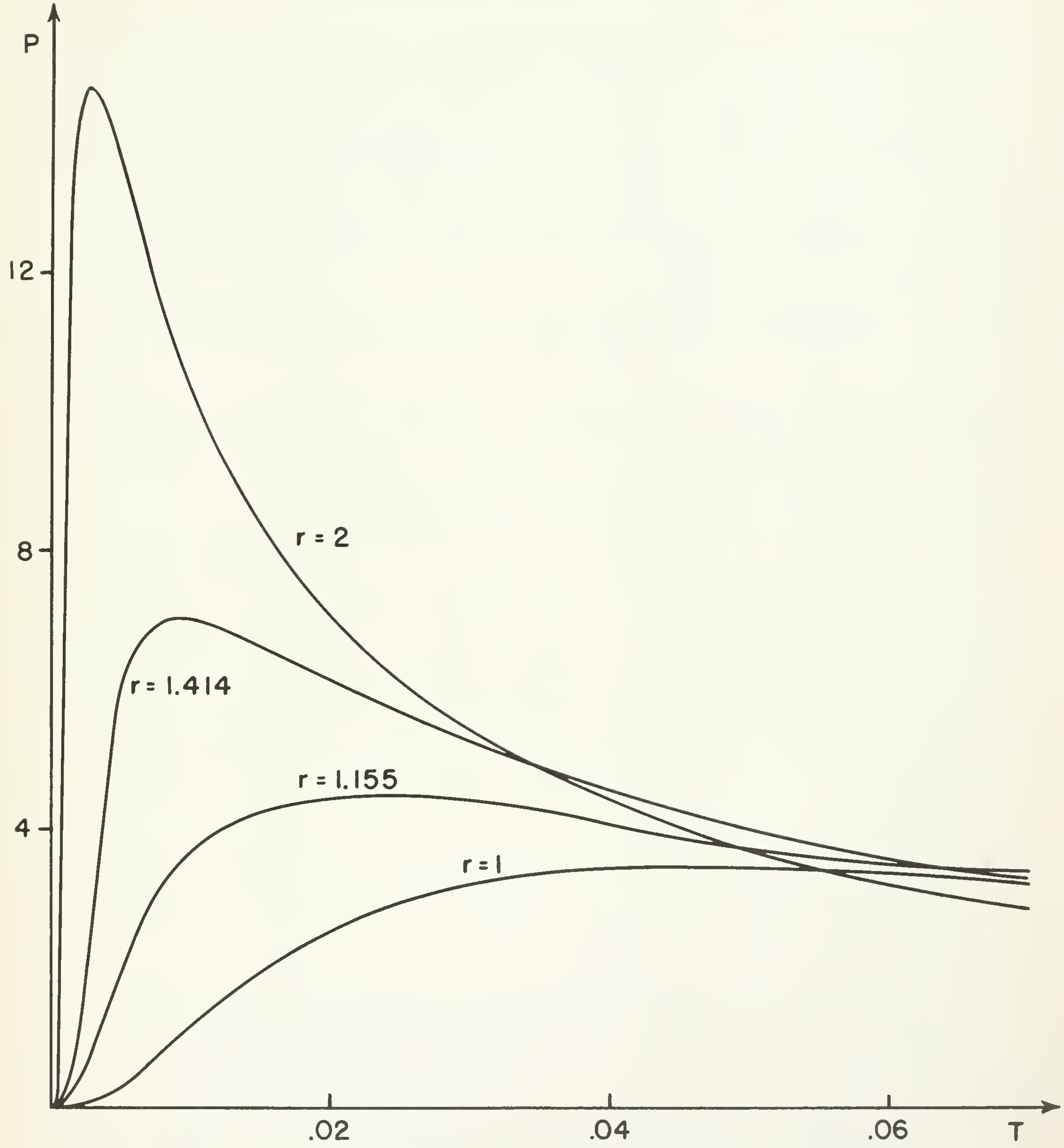

Figure 4

$$p(r,\theta,t) = \int_{-\infty}^{\infty} P(r,\theta,t-t') \, e^{i\omega t} \, dt'$$

but as $P(r,\theta,t-t') = 0$ for $t - t' < \sigma$ this is equivalent to

$$p(r,\theta,t) = e^{i\omega t} \int_{\sigma}^{\infty} P(r,\theta,t') e^{-i\omega t'} \, dt' = e^{i\omega t} \, \overline{P}(r,\theta,i\omega) \quad . \tag{9.2}$$

Hence

$$p(r,\theta,t) = e^{i\omega t} \sum_{m=-\infty}^{\infty} \overline{Q}(r,\theta + 2m\pi, i\omega) \quad . \tag{9.3}$$

Each term $\overline{Q}(r,\theta + 2m\pi, i\omega)$ of this series is a sum of certain modes, obtained by putting $s = i\omega$ in (.78). As $\omega \to \infty$, the asymptotic approximations can be used. For instance, when $r = 1$, a typical mode is asymptotically

$$\frac{3}{\sqrt{2}} \, \frac{e^{i\omega t}}{\alpha_j \, f(\alpha_j)} \quad \exp\left\{-\left[|\theta + 2m\pi| - \frac{\pi}{2}\right]\left[i\left(\omega + \frac{\alpha_j}{2}\,\omega^{1/3}\right) + \frac{3^{1/2}}{2}\,\alpha_j\omega^{1/3}\right]\right\} \quad . \tag{9.4}$$

This is a simple wave, propagated with phase velocity $(1 + \frac{1}{2}\,\alpha_j\omega^{-2/3})^{-1}$ and subject to an exponential attenuation which reduces the space of the amplitude by $1/e$ in the distance $3^{-1/2}\alpha_j^{-1}\omega^{-1/3}$. The attenuation increases both as ω and the order increases; also, if ω is large then the terms with $m > 0$, $m < -1$ in (9.3) are negligible. Because of the phenomenon mentioned at the end of Section 3, only the modes $j = 1$ can be used to estimate p for large ω. The resulting expressions are

$$\begin{aligned} p \sim {} & 1.837 \exp\left\{\frac{2\pi i}{\lambda}\left[t - (1+0.118\lambda^{2/3})\left(\theta - \frac{\pi}{2}\right)\right] - 1.295\lambda^{-1/3}\left(\theta - \frac{\pi}{2}\right)\right\} \\ & + 1.837 \exp\left\{\frac{2\pi i}{\lambda}\left[t - (1+0.118\lambda^{2/3})\left(\frac{3\pi}{2} - \theta\right)\right] - 1.295\lambda^{-1/3}\left(\frac{3\pi}{2} - \theta\right)\right\} \quad . \end{aligned} \tag{9.5}$$

Near $\theta = \pi$ these two terms are of comparable amplitudes, and as $|\theta - \pi|$ increases $|p|$ has maxima and minima approximately $\frac{1}{2}\lambda$ apart; but as $|\theta - \pi|$ increases further, one of the terms dominates the other one and $|p|$ increases exponentially. When λ is small enough, one may define an effective shadow boundary as the value of $\theta - \frac{\pi}{2}$ or as $\frac{3\pi}{2} - \theta$ at which $|p|$ equals a given small number ε, and assume that only one of the terms in (9.5) needs to be taken into account. The distance of this effective shadow boundary from the geometric shadow boundary is

$$(9.6) \qquad \theta - \frac{\pi}{2} = \frac{\lambda^{1/3}}{1.295} \log \frac{1.837}{\varepsilon}$$

that is to say this angle varies as the cube root of the wavelength.

Appendix A

Bessel Functions of Imaginary Order and Argument

1. In this appendix, an outline of the derivation of the approximations quoted in Section 3 will be given. The results are applied to Laplace transforms, which are necessarily regular in some half-plane $\mathrm{Re}(s) \geq c > 0$; hence it is sufficient to take s to be real and positive.

The differential equation satisfied by $K_{i\mu}(s)$ is

$$K''_{i\mu}(s) + \frac{1}{s} K'_{i\mu}(s) + \left(\frac{\mu^2}{s^2} - 1\right) K_{i\mu}(s) = 0 . \tag{A.1}$$

From this and the corresponding equation obtained by replacing μ by its complex conjugate μ^* it follows, unless μ is real, that

$$\int_s^{\infty} K_{i\mu}(s) K_{i\mu^*}(s) \frac{ds}{s} = \frac{s}{\mu^2 - \mu^{*2}} \left\{ K_{i\mu^*}(s) K'_{i\mu}(s) - K'_{i\mu^*}(s) K_{i\mu}(s) \right\} . \tag{A.2}$$

Now $K_{i\mu}(s)$ has the integral representation [8, p.181]

$$K_{i\mu}(s) = \int_0^{\infty} e^{-s \cosh \gamma} \cos \mu\gamma \, d\gamma . \tag{A.3}$$

Hence $K_{i\mu^*}(s)$ is the complex conjugate of $K_{i\mu}(s)$, and the left hand side of (A.2) is essentially positive. It follows that neither $K_{i\mu}(s)$ nor $K'_{i\mu}(s)$ can vanish for complex μ, as this would imply the vanishing of $K_{i\mu^*}(s)$ and $K'_{i\mu^*}(s)$ respectively and render (A.2) contradictory.

Thus the zeroes of $K'_{i\mu}(s)$, which are required, occur for real μ only, when s is real. Let μ be fixed, and s varied. When s is large then

$$K_{i\mu}(s) \sim \left(\frac{\pi}{2s}\right)^{1/2} e^{-s} > 0, \quad K'_{i\mu}(s) \sim -\left(\frac{\pi}{2s}\right)^{1/2} e^{-s} < 0 .$$

Hence as s is decreased from $+\infty$, $K_{i\mu}(s)$ is positive and increases at first. Let s_o be the greatest s such that $K'_{i\mu}(s) = 0$. Then $K_{i\mu}(s_o)$ is positive and a maximum, so that $K''_{i\mu}(s_o) = 0$. But by (A.1)

$$K''_{i\mu}(s_o) = \frac{\mu^2 - s_o^2}{s_o} K_{i\mu}(s_o)$$

and so necessarily $|\mu| \geq s_o$. This inequality remains true if s is fixed and μ is varied; hence the positive zeroes of $K'_{i\mu}(s)$, $\mu_1, \mu_2, \ldots$, all satisfy the inequality

(A.4) $$\mu_j \geq s \quad (j = 1, 2, \ldots) \ .$$

II. When s is large then by (A.4) the μ_j are also large, and $K_{i\mu_j}(s)$ can be approximated by Langer's method[5] (a convenient summary is given by Lighthill[6]). Let

(A.5) $$u(z) = K_{i\mu}(\mu z) \ ,$$

when z is real and positive. Then

(A.6) $$u''(z) + \frac{1}{z} u'(z) + \mu^2 \left(\frac{1}{z^2} - 1 \right) u(z) = 0 \ .$$

Hence for large μ, and $z < 1$,

(A.7) $$u(z) \sim \frac{\zeta^{1/2}}{(1 - z^2)^{1/4}} \left\{ A\, J_{1/3}(\zeta) + B\, J_{-1/3}(\zeta) \right\} \ ,$$

where A, B are independent of z, and

(A.8) $$\zeta = \int_z^1 \left(\frac{1}{z^2} - 1 \right)^{1/2} dz = \mu \cosh^{-1}(1/z) - \mu(1 - z^2)^{1/2} \ .$$

To determine A and B, let $z \to 0$. Then on the one hand

$$K_{i\mu}(\mu z) = \frac{\pi}{2 \sin(i\pi\mu)} \left\{ I_{-i\mu}(\mu z) - I_{i\mu}(\mu z) \right\} \sim \frac{1}{2i\mu} \left\{ (i\mu)!\left(\tfrac{1}{2}\mu z\right)^{-i\mu} - (-i\mu)!\left(\tfrac{1}{2}\mu z\right)^{i\mu} \right\}$$

whence

(A.9) $$u(z) \sim \left(\frac{2\pi}{\mu}\right)^{1/2} \exp\left(-\frac{1}{2}\pi\mu\right)\sin\left(\mu \log\frac{2}{ez} + \frac{\pi}{4}\right) .$$

On the other hand, $\zeta \sim \mu \log(2/ez)$ as $z \to 0$, and so (A.7) gives

$$u(z) \sim \left(\frac{2}{\pi}\right)^{1/2} \left\{A \cos\left(\mu \log\frac{2}{ez} - \frac{\pi}{4} - \frac{\pi}{6}\right) + B \cos\left(\mu \log\frac{2}{ez} - \frac{\pi}{4} + \frac{\pi}{6}\right)\right\} .$$

By comparing this with (A.9) one finds, after a short calculation, that

(A.10) $$u(z) \sim \frac{\pi \exp(-\frac{1}{2}\pi\mu)\zeta^{1/2}}{3^{1/2}\mu^{1/2}(1 - z^2)^{1/4}} \left\{J_{1/3}(\zeta) + J_{-1/3}(\zeta)\right\} .$$

Now let $z = s/\mu$, so that ζ is defined by

(A.11) $$\zeta = \mu \cosh^{-1}(\mu/s) - (\mu^2 - s^2)^{1/2} ;$$

then

(A.12) $$K_{i\mu}(s) \sim \frac{\pi \exp(-\frac{1}{2}\pi\mu)\zeta^{1/2}}{3^{1/2}(\mu^2 - s^2)^{1/4}} \left\{J_{1/3}(\zeta) + J_{-1/3}(\zeta)\right\} .$$

The fractional error of this approximation is $O(\mu^{-2/3})$, and since $\mu \geq s$ by hypothesis, this can be replaced by $O(s^{-2/3})$.

III. Let $s \to \infty$ in such a manner that ζ remains finite. Then $\zeta/s \to 0$, and by (A.11) this implies that $\mu/s \to 1$. When $(\mu/s)-1$ is small then

(A.13) $$\frac{\zeta}{s} = \frac{2^{3/2}}{3}\left(\frac{\mu}{s} - 1\right)^{3/2}\left\{1 + \dots\right\}$$

where the dots stand for a series in ascending powers of $(\mu/s)-1$. Hence $(\mu/s)-1$ is a series in ascending powers of $(\zeta/s)^{2/3}$. Let therefore

(A.14) $$\mu = s + \alpha s^{1/3} + O(s^{-1/3})$$

Then, by (A.13)

(A.15) $$\zeta = \frac{2^{3/2}}{3}\alpha^{3/2} + O(s^{-2/3})$$

and also

$$\mu^2 - s^2 = 2\alpha s^{4/3} + O(s^{2/3}) \quad .$$

When these approximations are substituted in (A.12) and in the expression by differentiating (A.12) with respect to s, one obtains

$$\text{(A.16)} \qquad K_{i\mu}(s) = \frac{2^{1/2}\pi}{3s^{1/3}} \exp(-\tfrac{1}{2}\pi\mu) f(\alpha) \left\{ 1 + O\left(s^{-2/3}\right) \right\}$$

$$\text{(A.17)} \qquad K'_{i\mu}(s) = -\frac{2^{1/2}\pi}{3s^{2/3}} \exp(-\tfrac{1}{2}\pi\mu) f'(\alpha) \left\{ 1 + O(s^{-2/3}) \right\}$$

where

$$\text{(A.18)} \qquad f(\alpha) = \alpha^{1/2} \left\{ J_{1/3}\left(\frac{2^{3/2}\ \alpha^{3/2}}{3}\right) + J_{1/3}\left(\frac{2^{3/2}\ \alpha^{3/2}}{3}\right) \right] \quad .$$

Hence the zeroes of $K'_{i\mu}(s)$ are given by

$$\text{(A.19)} \qquad \mu_j = s + \alpha_j s^{1/3} + O(s^{-1/3}) \quad ,$$

where $\alpha_1, \alpha_2, \ldots$ are the positive zeroes of $f'(\alpha)$, arranged in ascending order. This is true as $s \to \infty$ for fixed j; the constants implied by the O's depend on j, and presumably tend to infinity with j.

IV. It is now possible to derive approximations for the functions required in the evaluation of (2.16). In the first place, one may differentiate both sides of (A.17) with respect to α and then put $\mu = \mu_j$. This gives, because of (A.14),

$$\text{(A.20)} \qquad s^{1/3}\left[\frac{\partial}{\partial\mu} K'_{i\mu}(s)\right]_{\mu=\mu_j} = -\frac{2^{1/2}\pi}{3s^{2/3}} \exp(-\tfrac{1}{2}\pi\mu_j) f''(\alpha_j) \left\{ 1 + O(s^{-2/3}) \right\} \quad .$$

Now f(α) can be related to the Airy function by the equation

$$\text{(A.21)} \qquad f(\alpha) = \frac{3}{2^{1/6}} \operatorname{Ai}(-2^{1/3}\alpha)$$

and so

$$f''(\alpha) = 2\alpha f(\alpha) \quad . \tag{A.22}$$

Hence (A.20) can be replaced by

$$\left[\frac{\partial}{\partial\mu} K'_{i\mu}(s)\right]_{\mu=\mu_j} = \frac{2^{3/2}\pi}{3s} \alpha_j f(\alpha_j) \left\{1 + O(s^{-2/3})\right\} \quad . \tag{A.23}$$

Again, $K_{i\mu_j}(sr)$ is given by (A.12) with

$$\zeta = \zeta_j = \mu_j \cosh^{-1} \frac{\mu_j}{sr} - (\mu_j^2 - s^2r^2)^{1/2} \quad . \tag{A.24}$$

For fixed $r > 1$ and $s \to \infty$, $\mu_j < sr$ ultimately; hence it is convenient to replace (A.12) by an alternative formula. Now for $\mu_j < sr$,

$$\zeta_j = e^{3i\pi/2} \eta_j, \qquad \eta_j = (s^2r^2 - \mu_j^2)^{1/2} - \cos^{-1} \frac{\mu_j}{sr} \quad , \tag{A.25}$$

and so (A.12) can be written as

$$K_{i\mu_j}(sr) \sim \frac{\exp(-\frac{1}{2}\pi\mu_j)\eta_j^{1/2}}{(s^2r^2 - \mu_j^2)^{1/4}} K_{1/3}(\eta_j) \left\{1 + O(s^{-2/3})\right\} . \tag{A.26}$$

This is equivalent to (A.12) and either form is valid for all $r > 1$. But as (A.26) is too complicated for the evaluation of the inverse Laplace transform of a typical mode, one must approximate further, taking $r > 1$. Then for large s

$$\eta_j = s\left\{(r^2 - 1)^{1/2} - \cos^{-1} \frac{1}{r}\right\} - \alpha_j s^{1/3} \cos^{-1} \frac{1}{r} + O(s^{-1/3}) \ ,$$

and as this makes η_j large, one can put

$$\eta_j^{1/2} K_{1/3}(\eta_j) = (\tfrac{1}{2}\pi)^{1/2} \exp(-\eta_j) \left\{1 + O(\frac{1}{\eta_j})\right\} \quad .$$

Hence (A.26) can be replaced by the cruder approximation

$$K_{i\mu_j}(sr) \sim \frac{\pi^{1/2}}{(2s)^{1/2}(r^2-1)^{1/4}} \exp\left\{-\frac{1}{2}\pi\mu_j - s\left[(r^2-1)^{1/2} - \cos^{-1}\frac{1}{r}\right] + \alpha_j s^{1/3} \cos^{-1} \frac{1}{r}\right\} . \tag{A.27}$$

Appendix B

The Reflected Pulse

When the quantity δ defined by (3.10) is negative, the series expansions obtained by approximating the formula (2.16) for $\bar{F}$ do not converge, and it may be assumed that the evaluation of (2.10) by means of residues - which yields (2.16) - is not suitable for large s. Some other method of evaluating (2.10) must then be adopted. The simplest, and most satisfactory, case is $r = 1$. Then (2.10) reduces to

$$\bar{F}(1,\theta,s;r_0) = -\int_{-\infty}^{\infty} \frac{K_\nu(sr_0)}{sK_\nu'(s)} e^{i\nu\theta}\, d\nu \quad . \tag{B.1}$$

Now when s is large, real and positive, then $K_\nu(sr_0)/K_\nu'(s)$ is roughly independent of ν when $|\nu/s|$ is small; hence the principal contribution to the integral (B.1) arises from large values of $|\nu|$, and one may attempt to evaluate it by substituting appropriate asymptotic expressions for $K_\nu(sr_0)$ and $K_\nu'(s)$. These can be derived by applying Jeffreys' method (the WKB method) to the differential equation of K_ν, or from (A.26). Dropping the suffix and using the asymptotic formula for $K_{1/3}$, one can write this as

$$K_{i\mu}(s) \sim \frac{(\frac{1}{2}\pi)^{1/2}}{(s^2r^2 - \mu^2)^{1/4}} \exp(-\tfrac{1}{2}\pi\mu - \eta) \; .$$

Now let $\mu = i\nu$; then

$$\eta = \left(s^2r^2 + \nu^2\right)^{1/2} + i\nu \cos^{-1}\left(-\frac{i\nu}{sr}\right) = \left(s^2r^2 + \nu^2\right)^{1/2} - \nu \sinh^{-1}\frac{\nu}{sr} - \frac{1}{2} i\nu\pi \; ,$$

and so

$$K_\nu(sr) \sim \frac{(\frac{1}{2}\pi)^{1/2}}{(s^2r^2 + \nu^2)^{1/4}} \exp\left\{\nu \sinh^{-1}\frac{\nu}{sr} - \left(s^2r^2 + \nu^2\right)^{1/2}\right\} \; . \tag{B.2}$$

This is valid for complex ν if the imaginary part of $\sinh^{-1} \frac{\nu}{sr}$ is suitably restricted; if in particular ν is pure imaginary (and $|\nu| < sr$) then $-i \sinh^{-1} \frac{\nu}{sr} = \sin^{-1}(- \frac{i\nu}{sr})$ must be between $- \frac{1}{2}\pi$ and $+ \frac{1}{2}\pi$. Also

$$(B.3) \qquad sK_{\nu}'(s) \sim -(\tfrac{1}{2}\pi)^{1/2}(s^2+\nu^2)^{1/4} \exp\left\{\nu \sinh^{-1}\frac{\nu}{s} - (s^2+\nu^2)^{1/4}\right\} .$$

When (B.2) and (B.3) are substituted in (B.1) one obtains

$$(B.4) \qquad \bar{F}(1,\theta,s;r_o) \sim \int_{-\infty}^{\infty} e^{h(\nu)} g(\nu)d\nu$$

where

$$(B.5) \qquad h(\nu) = i\nu\theta + \nu \sinh^{-1}\frac{\nu}{sr_o} - \nu \sinh^{-1}\frac{\nu}{s} + (s^2+\nu^2)^{1/2} - (s^2r^2+\nu^2)^{1/2} ,$$

$$(B.6) \qquad g(\nu) = (s^2+\nu^2)^{-1/4}(s^2r^2+\nu^2)^{-1/4} .$$

One can now evaluate (B.4) approximately by the method of steepest descents. The function $h(\nu)$ has a saddle point at $\nu = \nu_o$, where

$$(B.7) \qquad i\theta + \sinh^{-1}\frac{\nu_o}{sr} - \sinh^{-1}\frac{\nu_o}{s} = 0 .$$

The solutions of this equation can be obtained geometrically, as follows. Let A be the point $(1,\theta)$ (where $\bar{F}$ is to be calculated) and B be the point $(r_o,0)$ (where the source is situated). If

$$(B.8) \qquad \alpha = \angle OBA \text{ and } \beta = \pi - \angle OAB, \text{ then } \theta = \beta - \alpha, \ \sin\beta = r_o \sin\alpha .$$

Hence (B.7) can be solved by putting

$$(B.9) \qquad \sinh^{-1}\frac{\nu_o}{s} = -i\beta, \quad \sinh^{-1}\frac{\nu_o}{sr_o} = i\alpha ,$$

and because of the restrictions $|\beta| < \frac{1}{2}$, $|\alpha| < \frac{1}{2}\pi$, this is the only admissible solution. Also, one must have $|\theta| < \cos^{-1}\frac{1}{r_o}$ that is to say $\delta < 0$.

At the saddle point,

$$h(\nu) = s(\cos\beta - r_0\cos\alpha) = -sR \tag{B.10}$$

where

$$R = \left(r_0^2 + 1 - 2r_0\cos\theta\right)^{1/2} \tag{B.11}$$

is the distance AB; also

$$h''(\nu_0) = -\frac{R}{sr_0\cos\alpha\cos\beta}, \qquad g(\nu_0) = \frac{1}{s\left(r_0\cos\alpha\cos\beta\right)^{1/2}} \quad . \tag{B.12}$$

Since $h''(\nu_0)$ is real and negative, the line of steepest descent through the saddle point is parallel to the real axis(it is assumed that s is real and positive), and hence

$$\bar{F}(1,\theta,s;r_0) \sim \left(\frac{2\pi sr_0\cos\alpha\cos\beta}{R}\right)^{1/2} \frac{e^{-sR}}{s\left(r_0\cos\alpha\cos\beta\right)^{1/2}} = \left(\frac{2\pi}{sR}\right)^{1/2} e^{-sR} \quad . \tag{B.13}$$

The inverse Laplace transform of this is

$$\bar{F}(1,\theta,t;r_0) \sim \left\{\frac{2}{R(t-R)}\right\}^{1/2} H(t-R) \quad . \tag{B.14}$$

Now the elementary solution G_0, defined by (1.10), is approximately given by

$$G_0(1,\theta,t;r_0) \sim \left\{\frac{1}{2R(t-R)}\right\}^{1/2} H(t-R) \tag{B.15}$$

at $(1,\theta)$, and so (B.14) can be written as

$$F(1,\theta,t;r_0) \sim 2G_0(1,\theta,t;r_0) \quad . \tag{B.16}$$

But this is equivalent to the well-known result that the pressure is doubled at the pulse front, on a reflector. One may therefore conclude that $F(1,\theta,t;r_0)$ represents the sum of the incident and reflected pulse.

If $r > 1$, and $\delta < 0$, one cannot proceed immediately in this manner. For it is now necessary to separate the incident and reflected pulses. It is not possible to do this at once with the expression (2.10) for $\bar{F}$, which is, for $r \leq r_0$,

$$\bar{F}(r,\theta,s;r_0) = \int_{-\infty}^{\infty} \frac{I_\nu(sr)K_\nu'(s) - I_\nu'(s)K_\nu(sr)}{K_\nu'(s)} K_\nu(sr_0)e^{i\nu\theta}\,d\nu \quad , \tag{B.17}$$

while r and r_o must be interchanged when $r \geq r_o$. For the resulting integrals would be divergent. But one may write (B.17) as the sum of two cosine transforms,

$$(B.18)\ \bar{F}(r,\theta,s;r_o) = 2\int_o^\infty I_\nu(sr)K_\nu(sr_o)\cos\nu\theta d\nu - 2\int_o^\infty \frac{I_\nu'(s)K_\nu(sr)K_\nu(sr_o)}{K_\nu'(s)}\cos\nu\theta d\nu\ .$$

The first term is obviously the elementary solution defined by (1.14) on the Riemann surface, when the cylinder is absent. Since this is the solution generated by an instantaneous source at $r = r_o$, $\theta = 0$, it follows from the uniqueness theorem that it will be identical with G_o for $0 < t < r_o$; when $t \geq r_o$, there will be an additional disturbance whose front is $t = r_o + r$. Thus it will be equal to G_o for small values of $t - R$ at any point of the reflection region, and the second term in (B.18) will be the additional disturbance due to the presence of the cylinder. This then is itself a solution of the wave equation, and reduces to $F - G_o \sim G_o$ on $r = 1$; it is a diverging pulse which vanishes for $t < R$ on the cylinder $r = 1$, and it must therefore have the reflected front as its pulse front. It is therefore the reflected pulse. As a matter of fact, if the Bessel functions are replaced by their asymptotic approximations, two integrals are obtained each of which can be estimated by the saddle-point method, and it is found that the sum of the contributions from the two saddle points is the leading term of the expression of the reflected pulse in half-integral powers of the time elapsed since the arrival of the reflected front.

REFERENCES

1. F.G. Friedlander, On the solutions of the wave equation with discontinuous derivatives, Proc. Camb. Phil. Soc. 38 (1943) 378 - 382.

2. F.G. Friedlander, Note on the geometrical optics of diffracted wave fronts, Proc. Camb. Phil. Soc. 45 (1949) 395 - 404.

3. J.B. Keller, Mechanics of Continuous Media, New York University Lectures, 1949-50.

4. D.E. Kerr, (Ed.), Propagation of Short Radio Waves, M.T.T. Rad. Lab. Series 13, New York 1951, Chapter 2.

5. R.E. Langer, On the asymptotic solutions of ordinary differential equations..., Trans. Amer. Math. Soc. 33 (1931) 23 - 64.

6. M.J. Lighthill, Reflection at a Laminar Boundary Layer..., Quart. Journ. Mech. and Applied Math. 3 (1950) 303 - 325.

7. J.C.P. Miller, The Airy Integral, Brit. Ass. Math. Tables, Part Volume B, Cambridge 1946.

8. G.N. Watson, Bessel Functions, Cambridge 1922.

The author also wishes to draw attention to a paper by Isao Imai (Z.für Physik, 137, 31-48, (1954), which has just appeared. In it the diffraction of plane harmonic electromagnetic waves by a circular cylinder is treated by a method different from the one presented here.

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Errata

Page 12: Replace the paragraph after Equation (3.16) by the following:

The question which now arises is to what extent both the series (2.16) and the approximations (3.9), (3.12), and (3.14) can be used. This has not been discussed in the present problem, but will be treated in the paper on propagation in a stratified atmosphere which has already been referred to. The results obtained there suggest that as $s \to \infty$, $\overline{F}$ can be replaced by the approximate expression (3.9), (3.12) or (3.14), for the first mode ($j=1$) of the series (2.16) provided that $\delta > 0$. Consequently, attention will from now on be confined to this term only.

Page 13: Second line of second paragraph:

read $sT - \alpha_1 \delta s^{1/3}$ instead of $sT - \alpha_1 \delta^{1/3}$.

Page 18: Sixth line of second paragraph:

read $F(r,\theta+2m\pi,t;r_o)$ instead of $F(r,\theta - 2m\pi,t;r_o)$

Page 29: Third line in first paragraph:

read $K''_{i\mu}(s_o) < 0$ instead of $K''_{i\mu}(s_o) = 0$

fourth line of first paragraph:

read $\dfrac{s_o^2 - \mu^2}{s_o^2}$ instead of $\dfrac{\mu^2 - s_o^2}{s_o}$.

Page 30: At the end of the paragraph before 'III' add the following:

The order terms given here and in III may differ from the correct ones. However, for the applications in this paper these differences are inconsequential.

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